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Inverse Method for Estimating Thermal Conductivity in One-Dimensional Heat Conduction Problems

Jiin-Hong Lin,* Cha'o-Kuang Chen,† and Yue-Tzu Yang‡
National Cheng-Kung University, Tainan 70101, Taiwan, Republic of China

An inverse analysis is provided to determine the spatial- and temperature-dependent thermal conductivities in several one-dimensional heat conduction problems. A finite difference method is used to discretize the governing equations, and then a linear inverse model is constructed to identify the undetermined thermal conductivities. The present approach is to rearrange the matrix forms of the differential governing equations so that the unknown thermal conductivity can be represented explicitly. Then, the linear least-squares-error method is adopted to find the solutions. The results show that only a few measuring points at discrete grid points are needed to estimate the unknown quantities of the thermal conductivities, even when measurement errors are considered. In contrast to the traditional approach, the advantages of this method are that no prior information is needed on the functional form of the unknown quantities, no initial guesses are required, and no iterations in the calculating process are necessary and that the inverse problem can be solved in a linear domain. Furthermore, the existence and uniqueness of the solutions can be easily identified.

Nomenclature

$\boldsymbol{A}$	=	coefficient matrix of vector $\boldsymbol{T}$
$\boldsymbol{B}$	=	coefficient matrix of vector $\boldsymbol{C}$
$\boldsymbol{C}$	=	vector constructed from the unknown thermal conductivities
$\boldsymbol{D}$	=	vector constructed from the functions of the unknown thermal conductivities
$\boldsymbol{E}$	=	product of $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$
$\boldsymbol{F}$	=	error function
g	=	heat generation, W/m^3
k	=	thermal conductivity, W/m $\cdot$ °C
q	=	heat flux, W/m^2
$\boldsymbol{R}$	=	reverse matrix
T	=	temperature, °C
$\boldsymbol{T}$	=	temperature vector
t	=	time, s
x	=	spatial coordinate, m
Δt	=	increment of time domain, s
Δx	=	increment of spatial coordinate, m
σ	=	standard deviation
ω	=	random variable

Subscripts

esti	=	estimated data
exact	=	exact data
i	=	index of spatial coordinate
j	=	index of time domain
meas	=	measured data
n	=	index at the boundary when x is equal to 1, m

Introduction

DURING the last decade, thermal property measurements have become important in modern engineering. Much effort has been devoted to studying inverse analysis in many design and manufacturing problems, especially when direct measurements of surface conditions are not possible. It is difficult to install thermocouples on the surface of heated objects. Therefore, if the temperature is measured at one or more interior locations of the objects and is then used to predict the surface thermal behavior of the objects, the difficulties encountered by Tseng et al.[1] with the surface mounting of thermocouples can be avoided. There have been numerous applications of inverse heat conduction problems (IHCP) in various branches of science and engineering, such as the prediction of the inner wall temperature of a reactor, the determination of the heat transfer coefficients and outer surface conditions of the space vehicle, and the prediction of temperature or heat flux at the tool–workpiece interface of machine cutting.

Inverse estimation of the thermal conductivity from temperature measurements has received great attention in recent years.[2–6] Many inverse problems are ill-posed, and a small measurement error may induce a large estimated error.[7–10] Hence, the solutions may not be unique. Most of the studies assumed that the thermal conductivity is only a function of spatial coordinates.[5,11–13] However, in many practical engineering applications, thermal conductivities are temperature-dependent quantities.[14] Various methods, including analytical or numerical approaches, have been developed to solve IHCP. Traditionally, the inverse problem includes two phases: the process of analysis and the process of optimization. In the analysis process, the unknown quantities are assumed, and then the results of the problem are solved directly through the numerical methods such as finite difference methods and finite element methods.[15,16] The solutions from the cited process are used to integrate with data measured at the interior point of the solid. Thus, a nonlinear problem is constructed for the process of optimization. In the optimization process, an optimizer, such as parameter estimation,[6] the least-squares method modified by the addition of a regularization term,[7,8] the conjugated gradient method,[5,17] and so on, is used to guide the exploring points systematically to search for a new set of guess quantities, which is then substituted for the unknown quantities in the analysis process. However, there are a few drawbacks in the traditional approaches. One is that the iterative process in the calculation cannot be avoided. The other is that the inverse problem can only be solved in a nonlinear domain.

To remedy the drawbacks in the traditional approaches, this work continues the study of a recently presented and innovative methodology[18] for solving IHCP. The method is applicable for reconstructing linear and nonlinear thermal conductivities, which can be either a constant or spatially dependent, as well as temperature-dependent quantities. The present approach is to rearrange the matrix forms of the differential governing equations such that the unknown

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*Graduate Student, Department of Mechanical Engineering.
†Professor, Department of Mechanical Engineering; ckchen@mail.ncku.edu.tw.
‡Professor, Department of Mechanical Engineering.

quantities can be represented explicitly. The linear least-squares-error method is then adopted to find the solutions of the unknown thermal conductivities. The inverse analysis is based on the temperature recordings taken within the medium at several different locations. In addition, no explicit functional form is assumed for the undetermined thermal conductivities, whereas in other methods polynomial or series forms are employed.

The advantages of this inverse method are that no prior information is needed on the functional form of the unknown quantities, no initial guesses are required, and no iterations in the calculating process are necessary and that the inverse problem can be solved in a linear domain. Furthermore, the existence and uniqueness of the solutions can be easily identified. However, the accuracy of the estimated thermal conductivity is sensitive to the errors of temperature measurement within the medium. In this work, the validity and robustness of the proposed methodology are presented and compared with the results of Yeung and Lam[19] in the first five examples.

Physical Model

Consider the problem concerning the inverse determination of the thermal conductivity for a heat conduction system. To compare the results from the current study with those from Yeung and Lam's[19] method, a one-dimensional, time-dependent nonhomogeneous model with heat generation is adopted. Figure 1 gives a general illustration of the medium, which is in the domain of $\{(x,t)|0 \le x \le 1\ \text{m}, t > 0\}$. The inverse determination of the thermal conductivity is based on the assumption that the initial condition $\{T(x,t)|t = 0\}$, the temperature distribution $\{T(x,t)|t > 0\}$, and the heat generation $\{g(x,t)|t \ge 0\}$ are known at all discrete grid points of the medium. The temperature outside the medium is assumed to be zero. Furthermore, the product of the material density and heat capacity is considered as unity.

The governing equation for the one-dimensional heat conduction problem can be expressed as

$$\frac{\partial T(x,t)}{\partial t} - \frac{\partial}{\partial x}\left[k(x,t)\frac{\partial T(x,t)}{\partial x}\right] = g(x,t)$$

$$0 < x < 1\ \text{m}, \qquad t \ge 0 \tag{1}$$

To satisfy the necessary condition[19] for the uniqueness of the thermal conductivity, the appropriate boundary conditions at $x = 0$ and 1 m can be classified as the following three types.

1) The first type of boundary condition specifies the temperature along the boundary surface at $x = 0$ or 1 m for $t > 0$:

$$T(x,t) \tag{2a}$$

2) The second type of boundary condition applies heat dissipation by convection from the boundary surface to a surrounding environment at zero temperature at $x = 0$ or 1 m for $t > 0$:

$$k(x,t)\frac{\partial T(x,t)}{\partial x} + T(x,t) \tag{2b}$$

3) The third type of boundary condition prescribes the heat flux at the boundary surface at $x = 0$ or 1 m for $t > 0$:

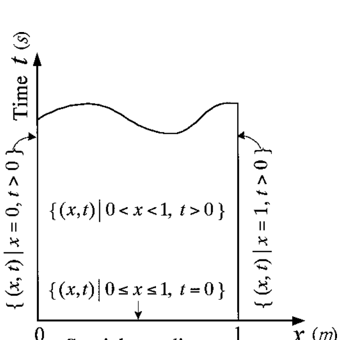


Fig. 1 Domain of the one-dimensional slab.

$$-k(x,t)\frac{\partial T(x,t)}{\partial x} \tag{2c}$$

Description of the Inverse Method

This inverse problem is to identify the unknown thermal conductivity $k(x,t)$ from the temperature measurements taken within the medium. For illustration, the implicit finite different method is employed to demonstrate the analysis process. After discretization, the governing equation [Eq. (1)] and the boundary conditions [Eqs. (2a–2c)] can be expressed in the following recursive forms.

The governing equation is

$$(T_{i,j} - T_{i,j-1})/\Delta t - (1/4\Delta x^2)[k_{i+1,j}(T_{i+1,j} - T_{i-1,j}) + 4k_{i,j}(T_{i+1,j} - 2T_{i,j} + T_{i-1,j}) + k_{i-1,j} \times (T_{i-1,j} - T_{i+1,j})] = g_{i,j} \tag{3}$$

The boundary conditions for $t > 0$ are as follows.

First type:

$$T_{0,j} \qquad \text{at} \qquad x = 0 \tag{4a}$$

$$T_{n,j} \qquad \text{at} \qquad x = 1\ \text{m} \tag{4b}$$

Second type:

$$k_{0,j}[(T_{1,j} - T_{0,j})/\Delta x] + T_{0,j} \qquad \text{at} \qquad x = 0 \tag{4c}$$

$$k_{n,j}[(T_{n-1,j} - T_{n,j})/\Delta x] + T_{n,j} \qquad \text{at} \qquad x = 1\ \text{m} \tag{4d}$$

Third type:

$$-k_{0,j}(T_{1,j} - T_{0,j})/\Delta x \qquad \text{at} \qquad x = 0 \tag{4e}$$

$$-k_{n,j}(T_{n-1,j} - T_{n,j})/\Delta x \qquad \text{at} \qquad x = 1\ \text{m} \tag{4f}$$

By substituting Eqs. (4a–4f) into Eq. (3) and rearranging the recursive forms consisting of the governing equation and the boundary conditions, we have an equivalent matrix equation that can be expressed as

$$\boldsymbol{AT} = \boldsymbol{D} \tag{5}$$

For the inverse model, matrix $\boldsymbol{A}$ can be constructed according to numerical methods and the known physical models, that is, the governing equation and the boundary conditions. Vector $\boldsymbol{T}$ is composed of the temperatures measured at several different locations within the medium by the thermocouples. The components of vector $\boldsymbol{D}$ are the functions of the unknown thermal conductivities.

The purpose of this research is to propose an approach to replace the nonlinear least-squares methods[2,20,21] so that the iterative calculations in the analysis and optimization phases can be eliminated in IHCP. In the proposed method, the linear inverse model [Eq. (5)] is constructed to represent the undetermined thermal conductivities explicitly. In the process of constructing the linear inverse model, no explicit functional form is assumed for the unknown thermal conductivities. Then, the unknown thermal conductivities and the differential form of heat conduction equation are rearranged. Finally, the measurements are substituted into the inverse model. As a result, the inverse model becomes a linear combination of the unknown thermal conductivities, and the inverse problems in Eq. (5) can then be solved through the linear least-squares-error method.

Decoupling the coefficients of the components of vector $\boldsymbol{D}$ will transform the linear inverse model [Eq. (5)] to the following inverse form:

$$\boldsymbol{AT} = \boldsymbol{BC} \tag{6}$$

where $\boldsymbol{D} = \boldsymbol{BC}$, $\boldsymbol{B}$ is the coefficient matrix of $\boldsymbol{C}$, and $\boldsymbol{C}$ is the vector of the unknown thermal conductivities at discrete grid points of the medium.

Assume that the estimated thermal conductivities $\boldsymbol{C}_{\text{esti}}$ can be obtained by means of the given estimated temperature $\boldsymbol{T}_{\text{esti}}$, that is,

$$\boldsymbol{A}\boldsymbol{T}_{\text{esti}} = \boldsymbol{B}\boldsymbol{C}_{\text{esti}} \tag{7a}$$

$$\boldsymbol{T}_{\text{esti}} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{C}_{\text{esti}} = \boldsymbol{E}\boldsymbol{C}_{\text{esti}} \tag{7b}$$

where $\boldsymbol{E} = \boldsymbol{A}^{-1}\boldsymbol{B}$.

Comparing the estimated temperature $\boldsymbol{T}_{\text{esti}}$ with the measured temperature $\boldsymbol{T}_{\text{meas}}$, we can represent the error function $\boldsymbol{F}$ as

$$\boldsymbol{F} = (\boldsymbol{T}_{\text{esti}} - \boldsymbol{T}_{\text{meas}})^T(\boldsymbol{T}_{\text{esti}} - \boldsymbol{T}_{\text{meas}}) \tag{8}$$

Substituting Eq. (7b) into Eq. (8), we can rewrite the error function $\boldsymbol{F}$ as the following matrix equation:

$$\begin{aligned}\boldsymbol{F} &= (\boldsymbol{E}\boldsymbol{C}_{\text{esti}} - \boldsymbol{T}_{\text{meas}})^T(\boldsymbol{E}\boldsymbol{C}_{\text{esti}} - \boldsymbol{T}_{\text{meas}}) \\ &= \left(\boldsymbol{C}^T_{\text{esti}}\boldsymbol{E}^T - \boldsymbol{T}^T_{\text{meas}}\right)(\boldsymbol{E}\boldsymbol{C}_{\text{esti}} - \boldsymbol{T}_{\text{meas}}) \\ &= \boldsymbol{C}^T_{\text{esti}}\boldsymbol{E}^T\boldsymbol{E}\boldsymbol{C}_{\text{esti}} - \boldsymbol{T}^T_{\text{meas}}\boldsymbol{E}\boldsymbol{C}_{\text{esti}} - \boldsymbol{C}^T_{\text{esti}}\boldsymbol{E}^T\boldsymbol{T}_{\text{meas}} + \boldsymbol{T}^T_{\text{meas}}\boldsymbol{T}_{\text{meas}}\end{aligned} \tag{9}$$

Then, the error function $\boldsymbol{F}$ can be minimized by differentiating $\boldsymbol{F}$ with respect to $\boldsymbol{C}_{\text{esti}}$ as

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{C}_{\text{esti}}} = 0 \tag{10}$$

Inserting Eq. (9) into Eq. (10), we have the following:

$$\boldsymbol{E}^T\boldsymbol{E}\boldsymbol{C}_{\text{esti}} + \boldsymbol{E}^T\boldsymbol{E}\boldsymbol{C}_{\text{esti}} - \boldsymbol{E}^T\boldsymbol{T}_{\text{meas}} - \boldsymbol{E}^T\boldsymbol{T}_{\text{meas}} = 0$$

$$\boldsymbol{E}^T\boldsymbol{E}\boldsymbol{C}_{\text{esti}} = \boldsymbol{E}^T\boldsymbol{T}_{\text{meas}} \tag{11}$$

Thus, vector $\boldsymbol{C}_{\text{esti}}$ can then be solved as follows:

$$\boldsymbol{C}_{\text{esti}} = (\boldsymbol{E}^T\boldsymbol{E})^{-1}\boldsymbol{E}^T\boldsymbol{T}_{\text{meas}} \tag{12}$$

where $(\boldsymbol{E}^T\boldsymbol{E})^{-1}\boldsymbol{E}^T$ is the reverse matrix of the inverse problem and is denoted as $\boldsymbol{R}$. The expressed process is derived by the linear least-squares-error method.

By estimating vector $\boldsymbol{C}$, the thermal conductivities $\{k(x,t)|0 \le x \le 1\text{ m}, t > 0\}$ at discrete grid points of the medium can be obtained simultaneously.

In Eq. (12), the inverse problem is solved by the linear least-squares-error method. As such, a special feature of this approach is that the iteration in the calculating process can be avoided, and the problem can be solved in a linear domain. Furthermore, it can be verified that the final solution, that is, Eq. (12), from the proposed method is the necessary condition of the optimum from the traditional nonlinear least-squares approach.[22] In the inverse problem, it is important to investigate the stability of the estimation. Usually, a small measurement error will induce a large estimated error in the ill-posed inverse problem. The methods of future time and regularization have been widely used to stabilize the results of the inverse estimation.[7,8] Those methods impose the physical condition onto the problem and increase the computational load in the estimated process. Consequently, the stability of the problem can be increased, while the computational load of the problem is also increased. In the present research, it is possible to stabilize the estimated results through a smooth process.[23] This method computes a moving average of the estimation. The result of data is the average of the N point around the current point. In this process, N must be an odd number. Then, the efficiency of the estimation can be increased.

In most cases, not all of the points need to be measured. The realistic experimental approach is to measure only a few points in the problems. Therefore, only parts of matrix $\boldsymbol{R}$, vector $\boldsymbol{T}$, and vector $\boldsymbol{C}$ corresponding to the measurement locations need to be constructed to estimate the unknown thermal properties of the inverse problem. However, in solving Eq. (12), the number of measurements needs to be sufficient so that the rank of the reverse matrix $\boldsymbol{R}$ is equal to the number of undetermined elements of vector $\boldsymbol{C}$. Otherwise, Eq. (12) will be underdetermined, and the problem cannot be solved through the proposed method. In general, when a large number of the measurements are selected, the costs for computation and experiment increase. However, the accuracy of the estimated results increases as well.

According to the described derivation, it is possible to recognize the existence and uniqueness of the solution. If the rank of reverse matrix $\boldsymbol{R}$ is less than the number of undetermined elements of vector $\boldsymbol{C}$, the number of measurements needs to be increased. Furthermore, if the rank of the reverse matrix $\boldsymbol{R}$ is equal to the number of undetermined elements of vector $\boldsymbol{C}$, the perpendicular distance from vector $\boldsymbol{C}$ to the column space of matrix $\boldsymbol{E}$ needs to be checked. If the distance vanishes, then the solution exists and is unique.

Results and Discussion

In this section, the heat conduction problems defined by Eqs. (1–2c) are used as the models for the inverse determination of thermal conductivity. Furthermore, the accuracy, efficiency, and versatility of the proposed inverse method will be verified in the following six examples. In the first two examples, the temperatures are prescribed along the boundary surfaces. The heat dissipation by convection from a surface to a surrounding environment at zero temperature is applied to the boundary condition in the third example. The heat fluxes specified at the left and right boundaries are illustrated in the last three examples.

The exact functions of temperature and thermal conductivity used in the following examples are preselected so that these functions can satisfy Eqs. (1–2c). The accuracy of the proposed method is verified by comparing the estimated results with the preselected profiles of exact thermal conductivity. The simulated temperature measurements are generated from the preselected exact temperatures, and they are presumed to contain measurement errors. In other words, the random errors of measurement are added to the exact temperatures. Thus, the simulated temperature measurement $\boldsymbol{T}_{\text{meas}}$ can be expressed as

$$\boldsymbol{T}_{\text{meas}} = \boldsymbol{T}_{\text{exact}} + \omega\sigma \tag{13}$$

where $\boldsymbol{T}_{\text{exact}}$ is the exact temperature, ω is the random error of the measurement, and σ is the standard deviation of the measurement error, which is assumed to be the same for all measurements. For normally distributed random errors, the probability of a random value ω lying in the range $-2.576 < \omega < 2.576$ is 99% (Ref. 24). In the present study, ω is considered as -2.576 or 2.576 for the most strict conditions.

In IHCP, the precision of the estimations at different measurement points depends strongly on the accuracy of the measurements. As will be seen in the examples, the estimated solutions that do not contain measurement error ($\sigma = 0$) converged to the exact solutions for all examples. Furthermore, the proposed method, with or without measurement error, yields unique solutions. Detailed descriptions for the inverse problems are shown in the following examples.

Example 1

In this example, a heat conduction problem for a slab, $0 \le x \le 1$ m, with constant thermal conductivity is presented. The slab, without considering the heat generation, is initially at a temperature $T(x,0) = \sin(\pi x)$, and the boundaries are subjected to zero temperatures at $t > 0$ as follows:

$$T(0,t) = 0 \tag{14a}$$

$$T(1,t) = 0 \tag{14b}$$

Moreover, the heat generation is assumed to be zero, and the preselected profiles of the exact temperature and thermal conductivity in the slab are

$$T(x,t) = e^{-2\pi^2 t}\sin(\pi x) \tag{14c}$$

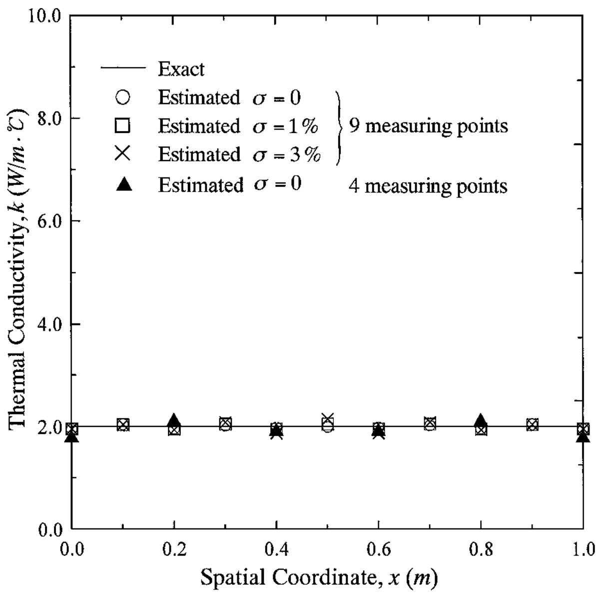


Fig. 2 Estimation of the thermal conductivities with measurement errors ($\sigma = 1$ and 3%) for $\Delta x = 0.1$ m and $t = 0.2$ s in example 1.

$$k(x, t) = 2 \tag{14d}$$

For this example, Fig. 2 shows a comparison between the estimated and exact thermal conductivities for mesh size $\Delta x = 0.1$ m and time $t = 0.2$ s. The effect of the number of points at which the temperature is measured on the accuracy of the results is also presented. From Fig. 2, we can see that whether the measurement errors ($\sigma = 1$ and 3%) are considered or not, the estimated results with nine measuring points are in good agreement with the exact thermal conductivities. It can also be seen in Fig. 2 that by using the proposed method, even with only four measuring points, a very good approximation of the predicted results is attained.

Example 2

A slab, $0 \le x \le 1$ m, with spatially dependent thermal conductivity is initially at zero temperature $T(x, 0) = 0$. The boundary conditions at both ends and the heat generation in the slab can be expressed as follows.

The boundary conditions at $t > 0$ are

$$T(0, t) = 0.36te^{-t} \tag{15a}$$

$$T(1, t) = 0.16te^{-t} \tag{15b}$$

The heat generation for $0 \le x \le 1$ m and $t > 0$ is

$$g(x, t) = (x - 0.6)^2(1 - t)e^{-t} - \left\{2 + [0.5 - 4(x - 0.3)(x - 0.6)]e^{-4(x-0.3)^2}\right\}te^{-t} \tag{15c}$$

The profiles of exact temperature and thermal conductivity in the slab are known in advance and given as

$$T(x, t) = (x - 0.6)^2te^{-t} \tag{15d}$$

$$k(x, t) = 1 + 0.25e^{-4(x-0.3)^2} \tag{15e}$$

A comparison of the inverse solutions with the exact ones for the spatial increment $\Delta x = 0.1$ m and time $t = 0.2$ s is shown in Fig. 3. In Fig. 3, without considering the measurement errors, the estimated values are very good approximations regardless of whether the number of measuring points are nine or four. Furthermore, even when the measurement errors ($\sigma = 1$ and 3%) are considered, the agreement between the predicted thermal conductivities and the exact ones is good. This indicates that the present method is more effective and accurate for IHCP.

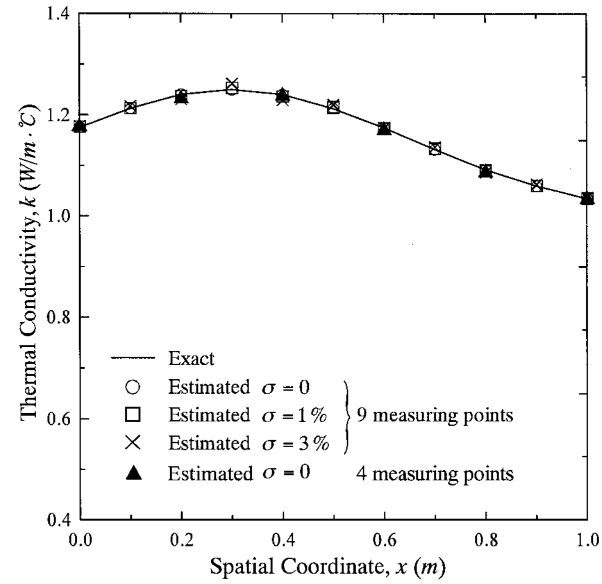


Fig. 3 Estimation of the thermal conductivities with measurement errors ($\sigma = 1$ and 3%) for $\Delta x = 0.1$ m and $t = 0.2$ s in example 2.

Example 3

An initial temperature $T(x, 0) = \cos\pi(x - 0.8)$ is applied to a slab, $0 \le x \le 1$ m, with temperature-dependent thermal conductivity. The time-dependent temperature gradient is prescribed at the boundary $x = 0$. In the meantime, the heat is dissipated at the boundary $x = 1$ m by convection into the environment of zero temperature. The boundary conditions at both ends and the heat generation in the slab can be written as follows.

The boundary conditions at $t > 0$ are

$$\left.\frac{\partial T(x, t)}{\partial x}\right|_{x=0} = \pi e^{-\pi^2 t}\sin 0.8\pi \tag{16a}$$

$$\left.\frac{\partial T(x, t)}{\partial x}\right|_{x=1} + T(1, t) = (\cos 0.2\pi - \pi\sin 0.2\pi)e^{-\pi^2 t} \tag{16b}$$

The heat generation at $t > 0$ is

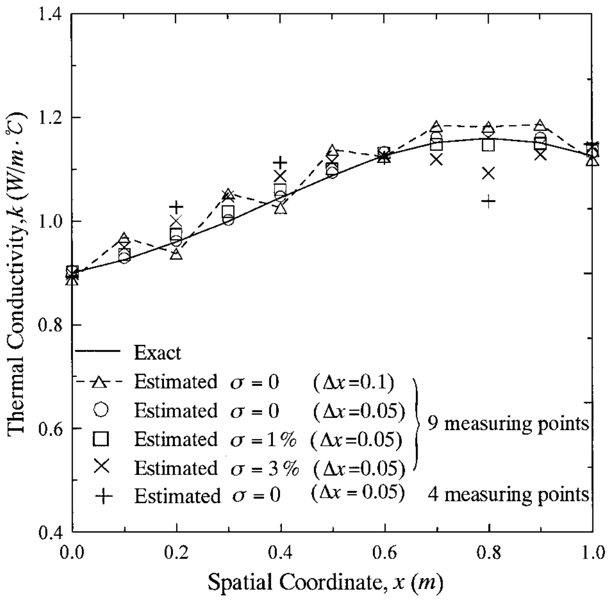
$$g(x, t) = -\pi^2 e^{-\pi^2 t}\cos\pi(x - 0.8) + \frac{\pi^2 e^{-\pi^2 t}\cos\pi(x - 0.8)}{1 - e^{-\pi^2 t}\cos\pi(x - 0.8)} - \left[\frac{\pi e^{-\pi^2 t}\sin\pi(x - 0.8)}{1 - e^{-\pi^2 t}\cos\pi(x - 0.8)}\right]^2 \tag{16c}$$

Moreover, the exact functions of temperature and thermal conductivity in the slab are preselected and given as

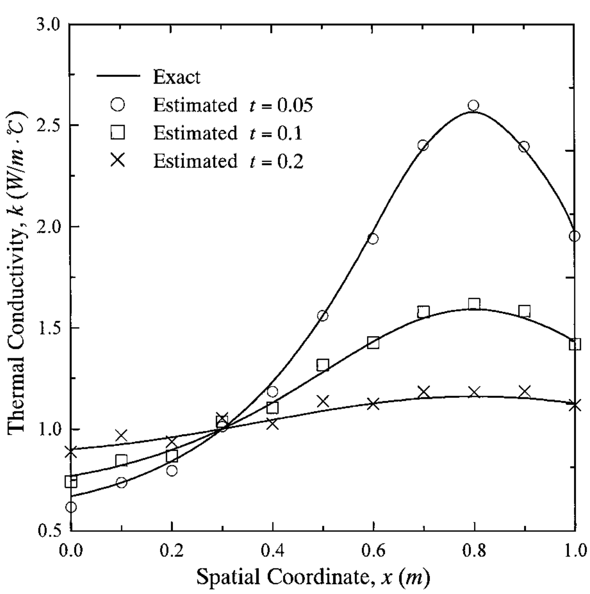
$$T(x, t) = e^{-\pi^2 t}\cos\pi(x - 0.8) \tag{16d}$$

$$k(x, t) = 1/[1 - T(x, t)] \tag{16e}$$

To predict the thermal conductivity and discuss the influence of the mesh size on the precision of the estimated results, the spatial coordinate $0 \le x \le 1$ m is divided into 10 and 20 intervals that correspond to the mesh size of $\Delta x = 0.1$ and 0.05 m, respectively. The estimated thermal conductivities for $\Delta x = 0.1$ and 0.05 m at time $t = 0.2$ s are shown in Fig. 4a. When the measurement errors are neglected, the estimated results for $\Delta x = 0.1$ m yields larger discrepancies than for $\Delta x = 0.05$ m. Thus, this implies that a decrease of mesh size increases the accuracy of the predicted thermal conductivity. However, when the measurement errors ($\sigma = 1$ and 3%) are considered, the magnitude of the discrepancies in the estimated thermal conductivity for $\Delta x = 0.05$ m is directly proportional to the measurement error. From Fig. 4a, we see that greater measurement error requires smaller mesh size to increase the accuracy of



a) With measurement errors ($\sigma = 1$ and 3%) for $\Delta x = 0.05$ and 0.1 m at $t = 0.2$ s



b) Without measurement errors for $\Delta x = 0.1$ m at $t = 0.05$, 0.1, and 0.2 s

Fig. 4 Estimation of the thermal conductivities in example 3.

the predicted results. Further, the effect of the number of points at which the temperature is measured on the accuracy of the results is also important. It can be seen in Fig. 4a that, for $\Delta x = 0.05$ m, the estimated results with nine measuring points are more accurate than those with four measuring points when measurement errors are not considered. Thus, the number of measuring points is required to be increased for more precise estimation of the thermal conductivities.

Without considering the measurement errors, the estimated and exact thermal conductivities at various times are shown in Fig. 4b for comparison. We see that the estimated values and the exact ones are in very good agreement, which also indicates that the unknown thermal conductivity can be predicted successfully and precisely through the inverse methodology presented in this work.

Example 4

A slab, $0 \le x \le 1$ m, with spatially dependent thermal conductivity, is initially at a temperature $T(x, 0) = (x - 3)^2$. The boundary surfaces at $x = 0$ and 1 m are both subjected to time-varying temperatures. Moreover, the heat flux is also known at the boundary $x = 0$.

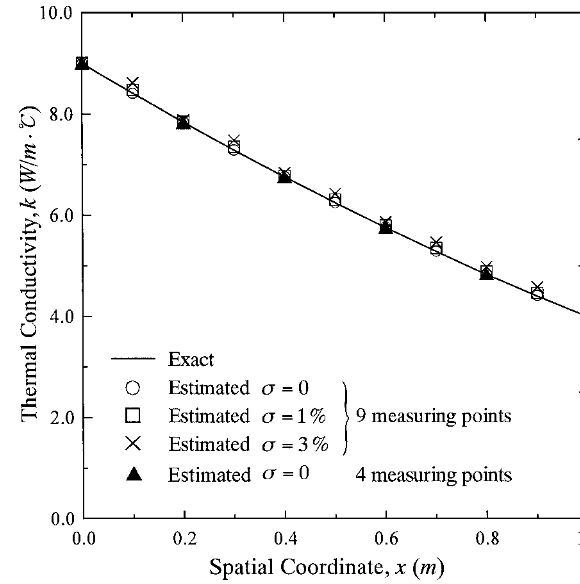


Fig. 5 Estimation of the thermal conductivities with measurement errors ($\sigma = 1$ and 3%) for $\Delta x = 0.1$ m and $t = 0.2$ s in example 4.

The boundary conditions at both ends and the heat generation in the slab can be shown as follows.

The boundary conditions at $t > 0$ are

$$T(0, t) = 9e^{-t} \tag{17a}$$

$$q(t)|_{x=0} = -54e^{-t} \tag{17b}$$

$$T(1, t) = 4e^{-t} \tag{17c}$$

The heat generation at $t > 0$ is

$$g(x, t) = -7(x - 3)^2 e^{-t} \tag{17d}$$

The preselected profiles of the exact temperature and thermal conductivity in the slab are

$$T(x, t) = (x - 3)^2 e^{-t} \tag{17e}$$

$$k(x, t) = (x - 3)^2 \tag{17f}$$

In this example, a comparison between the estimated and exact thermal conductivities is shown in Fig. 5 for the spatial increment $\Delta x = 0.1$ m and time $t = 0.2$ s. It can be seen in Fig. 5 that, whether or not the measurement errors ($\sigma = 1$ and 3%) are considered, a very good approximation is attained in the inverse analysis. Moreover, the results show that only four measuring points at discrete grid points are needed to estimate the thermal conductivities accurately.

Example 5

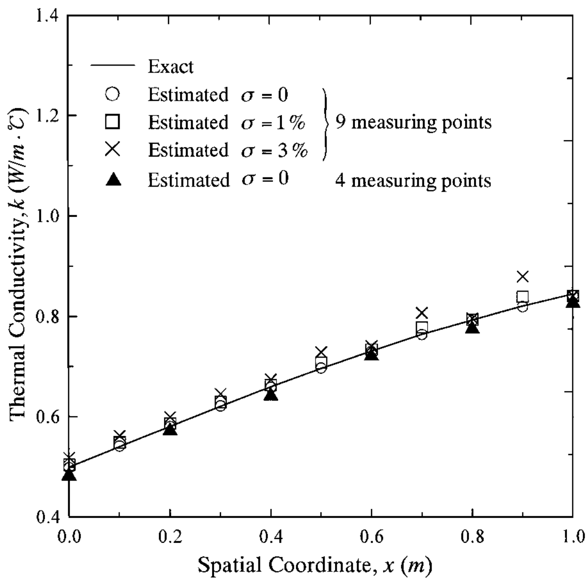
In this example, a slab, $0 \le x \le 1$ m, with temperature-dependent thermal conductivity is considered. The initial temperature of the slab is $T(x, 0) = \sin(x)$, and the boundary temperatures are specified at $x = 0$ and 1 m. In addition, the applied heat flux is prescribed at the boundary $x = 1$ m. Thus, the boundary conditions at both ends and the heat generation in the slab can be expressed as follows.

The boundary conditions at $t > 0$ are

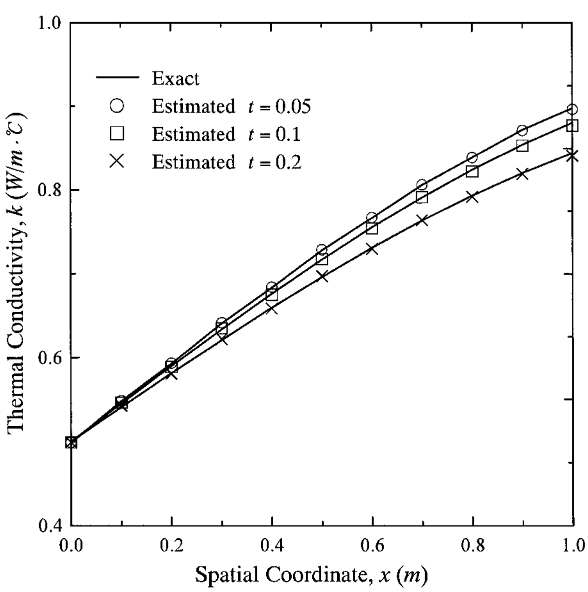
$$T(0, t) = 0 \tag{18a}$$

$$T(1, t) = e^{-t} \sin(1) \tag{18b}$$

$$q(t)|_{x=1} = 0.5[1 + e^{-t} \sin(1)]e^{-t} \cos(1) \tag{18c}$$



a) With measurement errors ($\sigma = 1$ and 3%) for $\Delta x = 0.1$ m at $t = 0.2$ s



b) Without measurement errors for $\Delta x = 0.1$ m at $t = 0.05, 0.1,$ and 0.2 s

Fig. 6 Estimation of the thermal conductivities in example 5.

The heat generation at $t > 0$ is

$$g(x, t) = -0.5e^{-t}\sin(x) - 0.5e^{-2t}[\cos^2(x) - \sin^2(x)] \quad (18d)$$

For this example, the exact temperature and thermal conductivity in the slab are preselected and known as

$$T(x, t) = e^{-t}\sin(x) \quad (18e)$$

$$k(x, t) = 0.5[1 + T(x, t)] \quad (18f)$$

Figure 6a shows a comparison of the inverse solutions with the exact ones for the spatial increment $\Delta x = 0.1$ m and time $t = 0.2$ s. It is obvious that the estimated thermal conductivity, without considering the measurement errors ($\sigma = 0$), and the exact profiles are in a good agreement when nine or four measuring points are taken. The estimated values are still good approximations even when measurement error $\sigma = 1\%$ is adopted. However, when considering the measurement errors $\sigma = 3\%$, slight discrepancies between the estimated and exact thermal conductivities are generated. In other words, it is clear that large measurement errors make the estimated results deviate from the exact values in the inverse analysis.

The estimated and exact thermal conductivities at various times are shown in Fig. 6b for comparison. In Fig. 6b, we see that the numerical results from this proposed method have accurate estimations when the measurement errors are not considered. Hence, it is quite obvious that the proposed inverse method is applicable and effective to deal with unsteady heat conduction problems.

Example 6

Consider a slab $0 \le x \le 1$ m, which is initially at a nonuniform temperature $T(x, 0) = (x + 1)^2$. The thermal conductivity in the slab is assumed to depend on temperature. For times $t > 0$, heat is generated within the solid at a rate of $g(x, t)$, while both the boundaries are subjected to the prescribed temperatures and heat flux that vary with time. The boundary conditions at both ends and the heat generation in the slab are given as follows.

The boundary conditions at $t > 0$ are

$$T(0, t) = e^{-t} \quad (19a)$$

$$q(t)|_{x=0} = 2e^{-t}(1 + 2e^{-t}) \quad (19b)$$

$$T(1, t) = 4e^{-t} \quad (19c)$$

$$q(t)|_{x=1} = 4e^{-t}(1 + 8e^{-t}) \quad (19d)$$

The heat generation at $t > 0$ is

$$g(x, t) = -12(x + 1)^2 e^{-2t} - [2 + (x + 1)^2]e^{-t} \quad (19e)$$

The preselected profiles of the exact temperature and thermal conductivity in the slab are

$$T(x, t) = (x + 1)^2 e^{-t} \quad (19f)$$

$$k(x, t) = 1 + 2T(x, t) \quad (19g)$$

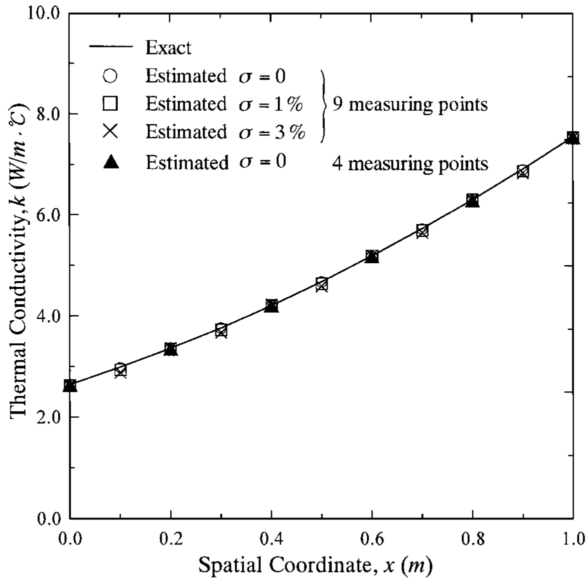
In this example, the estimated results for the spatial increment $\Delta x = 0.1$ m and time $t = 0.2$ s are shown. As shown in Fig. 7a, only four measuring points are required to estimate the results precisely when measurement errors are not considered. Also in Fig. 7a, it is shown that the estimated results are accurate and robust when measurement errors $\sigma = 1\%$ are included. Moreover, when the measurement error is $\sigma = 3\%$, the results are still satisfactory.

Figure 7b shows the comparison between the estimated and exact thermal conductivities at various times, when the measurement error is not considered. The results show that the estimated values have excellent agreement with the exact ones. Again, the presented inverse procedure is confirmed to be useful for solving unsteady heat conduction problems.

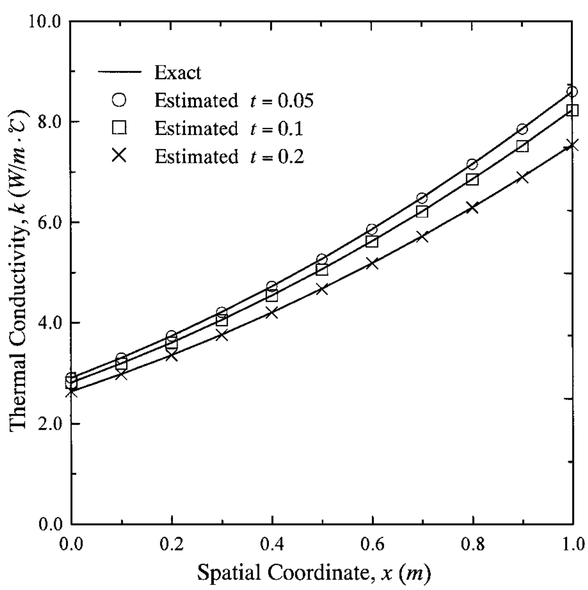
To discuss the influence of the mesh sizes on the precision of the estimated thermal conductivities, the spatial coordinate, $0 \le x \le 1$ m, is individually divided into 10, 20, and 40 intervals in the calculations for the preceding six examples. Without considering the measurement errors, the maximum errors on the estimated conductivities are given in Table 1 for mesh size $\Delta x = 0.1, 0.05$, and 0.025 m and time $t = 0.2$ s, which demonstrates that a decrease

Table 1 Maximum measurement errors on estimated thermal conductivities at $t = 0.2$ s

	Present study Δx, m			Yeung and Lam's study[19] Δx, m		
Example	0.1	0.05	0.025	0.1	0.05	0.025
1	0.0479	0.0122	0.0031	0.2575	0.0685	0.0174
2	0.0011	0.0007	0.0004	0.0073	0.0018	0.0004
3	0.0541	0.0221	0.0138	0.1031	0.0276	0.0070
4	0.0079	0.0056	0.0034	0.0385	0.0099	0.0025
5	0.0031	0.0022	0.0015	——	——	——
6	0.0077	0.0053	0.0032	——	——	——



a) With measurement errors ($\sigma = 1$ and 3%) for $\Delta x = 0.1$ m at $t = 0.2$ s



b) Without measurement errors for $\Delta x = 0.1$ m at $t = 0.05$, 0.1, and 0.2 s

Fig. 7 Estimation of the thermal conductivities in example 6.

of the mesh size reduces the maximum error on the estimated thermal conductivity. Also note that the maximum errors in the present study are much smaller than those in Yeung and Lam's study.[19] Consequently, the proposed method would seem to be superior and can provide more precise solutions for IHCP.

Conclusions

An inverse method has been successfully introduced for estimating the linear and nonlinear spatially dependent, as well as temperature-dependent, thermal conductivities in a one-dimensional slab. Using the reverse matrix, the proposed inverse model is reconstructed from the discrete forms of the differential heat conduction equation combined with the boundary conditions and the available temperature measurements. This inverse model can represent the unknown thermal conductivities explicitly, and the results can be solved without iteration by a linear least-squares-error method. Furthermore, the special feature of the proposed method is that the uniqueness of the solution can be easily identified. Six examples have been cited to confirm the applicability of the proposed method. The accuracy and the robustness of the present technique are verified by comparing the predicted results with the preselected analytical solutions. From the results, it appears that reasonably accurate estimations could be made even when measurement errors are considered. To increase the stability and accuracy of the predictions, the smaller mesh sizes in the calculations are required.

In contrast with the traditional approach, this proposed inverse analytic method requires no prior information on the functional form of the unknown quantities, requires no initial guesses, and requires no iterations in the calculating process. The advantages of this method are that the unknown quantities of the thermal conductivity can be estimated directly and that the inverse problem can be solved in a linear domain. Furthermore, the existence and uniqueness of the solutions can be easily identified. The presented inverse method is different from the traditional method using nonlinear least-squares formulation, which requires numerous iterations in the process and needs to be calculated in the nonlinear domain. This implies that the present model offers a great deal of flexibility. Consequently, the results show that the proposed method is an accurate, robust, and efficient inverse technique. Moreover, even though the presented inverse method is used to solve the one-dimensional IHCP, the method has the potential to be implemented in the field of multidimensional inverse heat conduction problems as well.

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References

[1]Tseng, A. A., Lin, F. H., Gunderia, A. S., and Ni, D. S., "Roll Cooling and Its Relationship to Roll Life," *Metallurgical Transactions A*, Vol. 20A, No. 11, 1989, pp. 2305–2320.

[2]Pfahl, R. C., Jr., "Nonlinear Least-Squares: A Method for Simultaneous Thermal Property Determination in Ablating Polymeric Materials," *Journal of Applied Polymer Science*, Vol. 10, No. 8, 1966, pp. 1111–1119.

[3]Alifanov, O. M., and Mikhailov, V. V., "Solution of the Nonlinear Inverse Thermal Conductivity Problem by the Iteration Method," *Journal of Engineering Physics*, Vol. 35, 1978, pp. 1501–1506.

[4]Tervola, P., "A Method to Determine the Thermal Conductivity from Measured Temperature Profiles," *International Journal of Heat and Mass Transfer*, Vol. 32, No. 8, 1989, pp. 1425–1430.

[5]Jarny, Y., Ozisik, M. N., and Bardon, J. P., "General Optimization Method Using Adjoint Equation for Solving Multidimensional Inverse Heat Conduction," *International Journal of Heat and Mass Transfer*, Vol. 34, No. 11, 1991, pp. 2911–2919.

[6]Ouyang, T., "Analysis of Parameter Estimation Heat Conduction Problems with Phase Change Using the Finite Element Method," *International Journal for Numerical Methods in Engineering*, Vol. 33, No. 10, 1992, pp. 2015–2037.

[7]Tikhnov, A. N., and Arsenin, V. Y., *Solutions of Ill-Posed Problems*, 1st ed., Winston, Washington, DC, 1977.

[8]Beck, J. V., Blackwell, B., and St. Clair, C. R., *Inverse Heat Conduction—Ill-Posed Problem*, 1st ed., Wiley, New York, 1985.

[9]Morozov, V. A., and Stessin, M., *Regularization Methods for Ill-Posed Problems*, CRC Press, Boca Raton, FL, 1993.

[10]Murio, D. A., *The Mollification Method and the Numerical Solution of Ill-Posed Problems*, Wiley, New York, 1993.

[11]Richter, G. R., "An Inverse Problem for the Steady State Diffusion Equation," *SIAM Journal on Applied Mathematics*, Vol. 41, No. 2, 1981, pp. 210–221.

[12]Kitamura, S., and Nakagiri, S., "Identifiability of Spatially-Varying and Constant Parameters in Distributed Systems of Parabolic Type," *SIAM Journal on Control and Optimization*, Vol. 15, No. 5, 1977, pp. 785–802.

[13]Lund, J., and Vogel, C. R., "A Fully-Galerkin Method for the Numerical Solution of an Inverse Problem in a Parabolic Partial Differential Equation," *Inverse Problems*, Vol. 6, No. 6, 1990, pp. 205–217.

[14]Artyukhin, E. A., "Recovery of the Temperature Dependence of the Thermal Conductivity Coefficient from the Solution of the Inverse Problem," *High Temperature*, Vol. 19, 1981, pp. 698–702.

[15]Bass, B. R., "Application of the Finite Element Method to the Nonlinear Inverse Heat Conduction Problem Using Beck's Second Method," *Journal of Engineering for Industry*, Vol. 102, No. 2, 1980, pp. 168–176.

[16]Beck, J. V., Litkouhi, B., and St. Clair, C. R., "Efficient Sequential Solution of Nonlinear Inverse Heat Conduction Problem," *Numerical Heat Transfer*, Vol. 5, No. 3, 1982, pp. 275–286.

[17]Alifanov, O. M., and Rumyantsev, S. V., "One Method of Solving Incorrectly Stated Problems," *Journal of Engineering Physics*, Vol. 34, No. 2, 1978, pp. 223–226.

[18]Yang, C. Y., and Chen, C. K., "The Boundary Estimation in Two-Dimensional Inverse Heat Conduction Problems," *Journal of Physics D: Applied Physics*, Vol. 29, No. 2, 1996, pp. 333–339.

[19]Yeung, W. K., and Lam, T. T., "Second-Order Finite Difference Approximation for Inverse Determination of Thermal Conductivity," *International Journal of Heat and Mass Transfer*, Vol. 39, No. 17, 1996, pp. 3685–3693.

[20]Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems," *Transactions of the American Society of Mechanical Engineers*, Vol. 82D, 1960, pp. 35–45.

[21]Sorenson, H. W., *Parameter Estimation: Principles and Problems*, Marcel Dekker, New York, 1980.

[22]Yang, C. Y., "Noniterative Solution of Inverse Heat Conduction Problems in One Dimension," *Communications in Numerical Methods in Engineering*, Vol. 13, No. 6, 1997, pp. 419–427.

[23]KaleidaGraph Reference Guide, Ver. 3.05, Abelbeck Software, 1994, p. 174.

[24]Silva Neto, A. J., and Ozisik, M. N., "Inverse Problem of Simultaneously Estimating the Timewise Varying Strength of Two-Plane Heat Source," *Journal of Applied Physics*, Vol. 73, No. 5, 1993, pp. 2132–2137.